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Corrections to the abelian Born-Infeld action arising from non-commutative geometry

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Abstract: In a recent paper Seiberg and Witten have argued that the full action describing the dynamics of coincident branes in the weak coupling regime is invariant under a specific field redefinition, which replaces the group of ordinary gauge transformations with the one of non-commutative gauge theory. This paper represents a first step towards the classification of invariant actions, in the simpler setting of the abelian single brane theory. In particular we consider a simplified model, in which the group of non-commutative gauge transformations is replaced with the group of symplectic diffeomorphisms of the brane world volume. We carefully define what we mean, in this context, by invariant actions, and rederive the known invariance of the Born-Infeld volume form. With the aid of a simple algebraic tool, which is a generalization of the Poisson bracket on the brane world volume, we are then able to describe invariant actions with an arbitrary number of derivatives.

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Contents

1. Introduction

The physics of branes with large background magnetic fields is intimately connected, as shown in various works[1] [1, 2], to gauge theories on non-commutative spaces. In particular, it has been shown, in a detailed study [1], that there exists a large freedom in the possible description of the physics of the gauge degrees of freedom which live on the brane world-volume. One is free to choose the non-commutativity parameter on the world-volume, and each possible choice can be reached by a suitable gauge-orbit preserving field redefinition. The most striking feature of the full action describing the brane dynamics at small string coupling is that, regardless of the choice of the non-commutativity parameter, it is, after the above-mentioned field redefinition, invariant in form, in a sense which will be made sharper in the later part of this introductory section. This property of the brane action is highly non trivial, and does constrain the action in a considerable but not fully understood way. In particular it has been argued in various settings [1, 7] that, if one considers only terms without derivatives, and if one looks at the U(1) single brane theory, then the unique action which is form invariant is the Born-Infeld one, which is known to describe the low energy phenomena of brane physics.

It is of importance to understand how the invariance described in [1] constrains the brane action in the more general non-abelian $\mathrm{U}(N)$ context, and also how it constrains higher derivative terms. There has been already some results in this di-

[1] For an extensive list of references, we refer the reader to [1].

rection [6], but the methods are not systematic, and become of increasing complexity after the first few terms have been constrained. This paper is a first step towards a classification of invariant actions, in a simplified context in which the geometric nature of the problem reduces the task to a manageable one. In particular we will not address the non-abelian case, restricting ourselves to the U(1) theory. Moreover we will work in a simplified setting, relying on a previous note [7, 8] by the author. In particular, we will substitute the group of non-commutative gauge transformations with the simpler group of symplectic diffeomorphisms of the brane world-volume, and we will carefully describe what we mean by invariant actions in this case. With the aid of a simple algebraic tool, which is a generalization of the natural Poisson bracket on the brane world-volume, we will then be able to generate in a simple and powerful way invariant actions with an arbitrary number of derivatives.

Let us note that the classification of invariant actions is intimately tied to a deeper understanding of T-duality in the context of open-string physics [9]. This can be better understood if we toroidally compactify space-time. It is then true that, for some integral values of the magnetic B-field, one can consider the brane configuration as a bound state of higher-dimensional branes with branes of lower dimension. T-duality then exchanges the two types of branes, and therefore also changes the underlying gauge group. There must therefore exist a (highly non-local) field redefinition which maps gauge orbits of one gauge group to gauge orbits of the other gauge group. Moreover the form of the action must be invariant under T-duality, and therefore the field-redefinition must respect the form of the action. Again, this is a highly non-trivial requirement, and it can be shown [9] to be equivalent, using simple Morita equivalence arguments, to the statements described in [1].

This paper has the following structure. We conclude this section by recalling the results of [1], both to set notation and to clarify what we mean by invariance of the brane action. In section 2 we then briefly recall the work [7] and describe the simplified setting within which we shall consider the problem. The invariance of the Born-Infeld action is then shown in section 3, and it is used to give a clear definition, in section 4, of what we mean by invariant actions within the setting of this paper. In section 5 we finally introduce the generalized bracket and we show how it can be used to construct invariant brane actions with an arbitrary number of derivatives. A few examples with two derivatives are then considered in section 6. Conclusions and open problems are left for the final section 7.

Let us then proceed to a quick review of [1]. We will work throughout with units such that

$$2\pi\alpha' = 1\,.$$

Let $M = \mathbb{R}^n$ be the flat space-time manifold, parametrized by coordinates x^a, and with constant background metric and NS two-form given by the matrices g_{ab} and B_{ab} (we will assume that B_{ab} is invertible). The arguments that follow do not rely on supersymmetry considerations, and are valid both in the context of bosonic string theory as well as in the context of type-II superstring theories. We then indicate with n the space-time dimension, with the understanding that $n = 26$ or $n = 10$.

We shall not be interested in the physics of the closed string sector, and we will accordingly leave the geometry of the background space-time manifold fixed. We will, on the other hand, concentrate on the dynamics of open string sector of the theory, by introducing N branes of maximal size — i.e. such that the brane world-volume coincides with the space-time manifold M. The dynamical degrees of freedom are then described by a $\mathrm{U}(N)$ connection on M. In the weak coupling regime $g_s \to 0$ the interaction of the brane gauge bosons are computed from string theory disk diagrams, and can be reconstructed from a low energy effective action of the general form

$$S = \frac{1}{g_s} \int d^n x \sqrt{\det g_{ab}}\, \mathrm{Tr}\left(1 + c^{abcd}\omega_{ab}\omega_{cd} + \cdots\right), \tag{1.1}$$

where

$$\omega = F + B$$

and the coefficients $c^{abcd}, \ldots$ are constructed from the tensor g_{ab} (for example the first coefficient is $\frac{1}{4}g^{ac}g^{bd}$). As indicated by the notation, the complete effect of the NS two-form B is obtained by replacing the $\mathrm{U}(N)$ field strength F with $\omega = F + B$ in the action.

The above action is defined only up to field redefinitions. The simplest type of redefinition, which had been already considered extensively in the works [3, 4, 5], are gauge covariant and leave the general form of the action invariant, with the unique effect of changing some of the coefficients. The redefinition is of the form $A_a \to A_a + d^{bc}D_bF_{ac} + \cdots$, where again the coefficients $d^{bc}, \ldots$ are constructed in terms of the metric. A more powerful possible field redefinition has been shown to exist in the recent work [1]. The change of variables does not preserve the group of gauge transformations, but on the other hand it substitutes it with the group of gauge transformations of non-commutative gauge theory on the world-volume M. More precisely, there is a transformation $A_a \to \widehat{A}_a$ (which we shall call Seiberg-Witten transformation) preserving gauge orbits such that, in terms of the non-commutative field strength $\widehat{F}$, or, better, of the combination

$$\Omega = \widehat{F} - B\,,$$

the action reads

$$S = \frac{1}{G_s} \int d^n \sigma \sqrt{\det G_{ab}}\, \mathrm{Tr}\left(1 + C^{abcd}\Omega_{ab} \star \Omega_{cd} + \cdots\right).$$

In the above, the new metric tensor G_{ab} and string coupling constant G_s are given by

$$G = -B\frac{1}{g}B\,, \qquad \frac{1}{g_s}\sqrt{\det B} = \frac{1}{G_s}\sqrt{\det G}\,. \tag{1.2}$$

Moreover, the coefficients C^{abcd} are obtained starting from the coefficients c^{abcd} and replacing the metric g_{ab} with G_{ab}. Finally, the non-commutativity parameter defining the product $\star$ and the field-strength $\widehat{F}$ is given by[2]

$$\theta = \frac{1}{B}\,.$$

In the work [1] the transformation $A_a \to \widehat{A}_a$ is determined using the two requirements that it must preserve gauge orbits and that it must be expressible as a power series in θ, with the coefficients of the series being local expressions in the fields. These requirements clearly defines the map up to gauge-covariant local field redefinitions. On the other hand a more precise statement of [1] says that there is, among the possible maps $A_a \to \widehat{A}_a$, one for which the action is form-invariant, in the sense described above.

The problem of invariance has not been analyzed in any detail. We will now describe, in the next sections, a simplified setting which will allow us to tackle the problem in a simple but powerful way.

2. The simplified setting

We will, throughout the rest of this paper, work in the simplified context of the abelian U(1) theory. Although this choice does imply a considerable loss of information, we will see that the abelian theory has already a rich structure, and does provide partial information about the non-abelian case.

The second simplification concerns the map $A_a \to \widehat{A}_a$, and follows the author's previous note [7]. In this section we quickly review the results of [7], and we rephrase them in the context of the problem at hand. The starting point of [7] is the observation that the two-form $\omega = B + F$ defines a symplectic structure on M. On one hand ω is clearly closed. Moreover, since we always work perturbatively in F, and since B is invertible, one can take the formal inverse of ω, thus showing that ω is non degenerate. Therefore, by Darboux's theorem, one can find coordinates σ^i on M such that[3]

$$\omega = \frac{1}{2} B_{ij}\, d\sigma^i \wedge d\sigma^j\,.$$

In these new coordinates, the fluctuations of the field strength F have been replaced by the parallel displacements of the brane, which are described by the coordinate

[2] We recall that $f \star g = \exp(\frac{i}{2}\theta^{ij}\partial_i^f \partial_j^g) f \cdot g$ and that $\widehat{F}_{ab} = \partial_a \widehat{A}_b - \partial_b \widehat{A}_a - i\widehat{A}_a \star \widehat{A}_b + i\widehat{A}_b \star \widehat{A}_a$.

[3] We will use the following general conventions concerning coordinate systems and indices. A general coordinate system on M will be denoted by ξ^α, and in general will have Greek indices $\alpha, \beta, \ldots$ The fixed coordinate system x^a will be called flat, and will have in general roman indices $a, b, \ldots$ Finally coordinate systems σ^i for which the two-form ω has constant coefficients B_{ij} will be called symplectic, and will have roman indices $i, j, \ldots$

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functions $x^a(\sigma)$. Moreover the coordinates σ^i are clearly defined up to symplectic diffeomorphisms of M. The original group of abelian gauge transformations is replaced by the group of symplectomorphisms of (M,ω), and the correspondence between $A_a(x)$ and $x^a(\sigma)$ respects the gauge orbits of the two group actions. One is therefore in a situation similar to the one considered in [1], with the simplifying difference that the group of non-commutative gauge transformations is replaced by the group of symplectomorphisms of the brane world-volume.

To make contact with the notation of the previous section one defines the Poisson bracket $\{\,,\}$ with respect to the symplectic structure on M

$$\{f,g\} = \left(\frac{1}{\omega}\right)^{\alpha\beta} \partial_\alpha f\, \partial_\beta g\,.$$

The above formula is particularly simple if one uses the symplectic σ-coordinates, and it then reads

$$\{f,g\} = \theta^{ij}\partial_i f\, \partial_j g\,.$$

If one defines the non-commutative gauge potential $\widehat{A}_a$ by

$$x^a(\sigma) = \sigma^a + \theta^{ab}\widehat{A}_b(\sigma)$$

and the corresponding field strength by[4]

$$\widehat{F}_{ab} = \partial_a \widehat{A}_b - \partial_b \widehat{A}_a + \left\{\widehat{A}_a, \widehat{A}_b\right\},$$

one finds that

$$\{x_a, x_b\} = \Omega_{ab} = \widehat{F}_{ab} - B_{ab}\,,$$

where we have lowered the index on the coordinate function x^a using B_{ab}

$$x_a = B_{ab}x^b\,.$$

We have quickly reviewed the results of [7] and we are therefore in a position, given the above notation, to rephrase the meaning of the invariance of the action given in section 1 in this new framework, starting from the simple invariance of the Born-Infeld volume form.

3. Invariance of the Born-Infeld volume form

In this section we prove the exact invariance of the Born-Infeld volume form under the change of coordinates described in the previous section. Before we do so, let us though clarify one point of notation.

[4]This is clearly an exception to the index convention, which is forced by the notation.

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In order to limit the number of symbols, we will use, as a general rule, the same letter to indicate an abstract tensor, and its components in a specific coordinate system, and we will rely on our index convention to distinguish among coordinate systems. In some cases though this might be confusing, given the standard notation in the subject. For example the metric tensor $g = g_{ab}dx^a \otimes dx^b$ reads, in a symplectic coordinate system (recall $G_{ab} = B_{ac}B_{bd}g^{cd}$)

$$g = g_{ab}\partial_i x^a \partial_j x^b \, d\sigma^i \otimes d\sigma^j = G^{ab}\partial_i x_a \partial_j x_b \, d\sigma^i \otimes d\sigma^j \,.$$

Following the general rule, we could use the symbol g_{ij} for $G^{ab}\partial_i x_a \partial_j x_b$. We will not do this, and we will reserve the letter g_{ab} for the constant metric in the flat coordinate system, and will always write $G^{ab}\partial_i x_a \partial_j x_b$ when using σ-coordinates. Similarly we will use B_{ij} instead of ω_{ij}.

With this in mind, let us consider the Born-Infeld volume form

$$\Phi = \frac{1}{g_s} d^n\xi \sqrt{\det(g+B+F)} = \frac{1}{g_s} d^n\xi \sqrt{\det(g+\omega)} \,.$$

In flat coordinates x^a one has

$$\Phi = \frac{1}{g_s} d^n x \sqrt{\det(g+\omega)_{ab}} \,.$$

Let us now compute the volume form Φ in the symplectic coordinates σ^i. If we introduce the matrix-valued field

$$M^i{}_j = \theta^{ik} G^{ab} \partial_k x_a \partial_j x_b$$

we easily compute that

$$\begin{aligned}
\mathrm{Tr}\, M &= G^{ab}\Omega_{ba} = \mathrm{Tr}\, \frac{1}{G}\Omega \,, \\
\mathrm{Tr}\, M^2 &= G^{ab}G^{cd}\, \mathrm{Tr}\, \theta\partial x_a \partial x_b \theta \partial x_c \partial x_d = G^{ab}\Omega_{bc}G^{cd}\Omega_{da} = \mathrm{Tr}\, \frac{1}{G}\Omega\frac{1}{G}\Omega \,, \\
\mathrm{Tr}\, M^n &= \mathrm{Tr}\left(\frac{1}{G}\Omega\right)^n \,.
\end{aligned}$$

Using the above equation one can show that

$$\sqrt{\det(1+\theta G^{ab}\partial x_a \partial x_b)} = \sqrt{\det\left(1+\frac{1}{G}\Omega\right)}$$

by expanding $\det^{1/2}(1+A)$ in terms of traces of powers of A. We may then check that, in the symplectic coordinates σ^i, the Born-Infeld form reads (using eq. (1.2))

$$\begin{aligned}
\Phi &= \frac{1}{g_s} d^n\sigma \sqrt{\det(B + G^{ab}\partial x_a \partial x_b)} = \frac{1}{G_s} d^n\sigma \sqrt{\det G(1+\theta G^{ab}\partial x_a \partial x_b)} \\
&= \frac{1}{G_s} d^n\sigma \sqrt{\det G\left(1+\frac{1}{G}\Omega\right)} = \frac{1}{G_s} d^n\sigma \sqrt{\det(G+\Omega)} \,,
\end{aligned}$$

thus proving the invariance of the Born-Infeld action under the simplified Seiberg-Witten change of variables described in the previous section.

4. Statement of the invariance problem

We now have the notation needed to define the concept of invariant action within the setting of this paper, and to describe the problem that we wish to address. We will not give general definitions and proofs, but we will work with some meaningful examples, with the hope that the general case can be easily understood from them.

We start with a basic observation, by noting that

$$\{\sigma^i, f\} = \theta^{ij}\partial_j f\,,$$

and therefore that

$$\{x_a, f\} = \partial_a f + \{\widehat{A}_a, f\} \equiv \widehat{D}_a f\,.$$

Then, in the limit $B \to \infty$, $\theta \to 0$, one has that[5]

$$\{x_a, f\} \longrightarrow \partial_a f\,.$$

This means that the correct dual for a generic derivative term

$$\partial_a\partial_b\omega_{cd}$$

is given by

$$\{x_a, \{x_b, \Omega_{cd}\}\} = \{x_a, \{x_b, \{x_c, x_d\}\}\}\,.$$

At the level of the action S describing (eq. (1.2))the brane world-volume degrees of freedom we may consider a generic term

$$\frac{1}{g_s}\int d^n x\sqrt{\det g_{ab}}\, g^{ad}g^{be}g^{cf}\partial_a\omega_{bc}\partial_d\omega_{ef}\,.$$

Following the above observations, this term in the action should correspond, in the symplectic coordinates σ^i, to the term

$$\frac{1}{G_s}\int d^n\sigma\sqrt{\det G_{ab}}\, G^{ad}G^{be}G^{cf}\{x_a, \Omega_{bc}\}\{x_d, \Omega_{ef}\}\,.$$

In the above discussion we have not addressed an important issue, which depends on the fact that we are not considering the full non-abelian theory, but that we are uniquely concentrating on the abelian U(1) theory, and that we are therefore neglecting commutator terms. More precisely, let us note, for example, that the correspondence

$$\partial_a\partial_b\omega_{cd} \longrightarrow \{x_a, \{x_b, \Omega_{cd}\}\}$$

[5] The correspondence $\{x_a, = \widehat{D}_a \to \partial_a$ is a notable exception to the index convention, since we are using an index $a, b, \ldots$ in a symplectic coordinate system. This is the same exception noted in the previous footnote, since $\widehat{F}_{ab} = \widehat{D}_a\widehat{D}_b - \widehat{D}_b\widehat{D}_a$.

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is not well-defined. In fact, although the partial derivatives $\partial_a \partial_b$ commute, the corresponding Poisson bracket derivatives $\{x_a, \{x_b, \ldots\}\}$ do not. On the other hand, the commutator is proportional, by the Jacobi identity, to

$$\{\Omega_{ab}, \ldots\}$$

and therefore vanishes in the limit $B \to \infty$, $\theta \to 0$. We can therefore state a precise form of the invariance concept which we wish to analyze.

Definition 1 *The abelian action S for a single brane is invariant under the simplified Seiberg-Witten transformation described in section 2 if it has, when written in terms of the symplectic coordinates σ^i, the same form (as described in this section) as in the original flat coordinates x^a, up to terms which vanish in the $B \to \infty$, $\theta \to 0$ limit.*

In the next section we will see that, with the aid of a simple algebraic tool, we will be able to easily generate actions which do possess the property just described.

5. The generalized bracket and invariant actions

In order to construct in a systematic way actions which are invariant in the sense described above, we introduce a bilinear differential operation defined on functions, which generalizes the Poisson bracket $\{\,,\}$. Given two functions f and g on M we define the bracket $[f, g]$ by

$$[f, g] = \left(\frac{1}{g+\omega}\right)^{\alpha\beta} \partial_\alpha f \, \partial_\beta g \, .$$

Let me note that the above bilinear form is not antisymmetric

$$[f, g] \neq -[g, f]$$

and does not satisfy the Jacobi identity. On the other hand it will be an extremely useful tool in constructing invariant actions.

An intuitive argument for the above definition is the following. We know that the effect of the NS two-form B is described with the replacement $F \to \omega = F + B$. This is justified by looking at the string conformal field theory $\int_\Sigma B + \int_{\partial\Sigma} A$ (Σ is the string world-volume) and by noting that the transformation $B \to B + d\Lambda$, $A \to A - \Lambda$, leaves both the action and ω invariant. On the other hand, if one considers the massless closed string sector vertex operators $(h_{ab} + B_{ab}) \int d^2z \, \partial X^a \overline{\partial} X^b e^{ik\cdot X}$ one notices that the natural combination[6] which appears is $g + B$. Therefore, one is lead to look at expression $g + B + F = g + \omega$, which is present both in the Born-Infeld volume form, and in the bracket $[\,,]$ which we have just described.

[6]The combination $E = g + B$ is also relevant in T-duality on tori, where it transforms as $E \to E^{-1}$.

Let us start with a first basic example, by computing the bracket $[x^a, x^b]$. In the flat coordinates x^a one simply obtains

$$[x^a, x^b] = \left(\frac{1}{g+\omega}\right)^{ab}.$$

On the other hand one can compute the bracket $[\,,]$ in the symplectic coordinates σ^i. In this case it will be easier to consider the quantity $[x_a, x_b]$, where we have lowered the indices with $x_a = B_{ab}x^b$. From the definition we obtain that

$$[x_a, x_b] = \left(\frac{1}{\partial x_c \partial x_d G^{cd} + B}\right)^{ij} \partial_i x_a \, \partial_j x_b \,.$$

We can then expand the denominator in powers of the induced metric $\partial x_c \partial x_d G^{cd}$ and obtain

$$\begin{aligned}
[x_a, x_b] &= \left(\theta - \theta \partial x_c \partial x_d G^{cd} \theta + \cdots\right)^{ij} \partial_i x_a \, \partial_j x_b \\
&= \{x_a, x_b\} - \{x_a, x_c\} G^{cd} \{x_d, x_b\} + \cdots \\
&= \left(\Omega - \Omega \frac{1}{G} \Omega + \cdots\right)_{ab} \\
&= \left[G \left(\frac{1}{G} - \frac{1}{G+\Omega}\right) G\right]_{ab}.
\end{aligned}$$

For reasons which we will shortly describe, we define the functions y^a by

$$y^a = G^{ab} x_b = G^{ab} B_{bc} x^c \,.$$

We can then use the previous computation and write

$$[y^a, y^b] = \left(\frac{1}{G} - \frac{1}{G+\Omega}\right)^{ab}. \tag{5.1}$$

The importance of the functions y^a is that they play, in a symplectic coordinate system, the same role played by the coordinates x^a in a flat coordinate system. More precisely, all contractions of the functions x^a with metric tensors g_{ab} are equal to equivalent contractions of the functions y^a with the tensor $G_{ab} = B_{ac}B_{bd}g^{cd}$. For example

$$g_{ab}\left[x^a, x^b\right] = G^{ab}[x_a, x_b] = G_{ab}[y^a, y^b]\,.$$

We can then consider the simple action

$$\int \Phi \, g_{ab}\left[x^a, x^b\right] = \int \Phi \, G_{ab}[y^a, y^b]\,.$$

We have proved in the last section the invariance of the Born-Infeld volume form. If one neglects the term $(1/G)^{ab}$ and the minus sign in the expression 5.1 for $[y^a, y^b]$, the previous computation would shows that the above action is invariant. Clearly though one cannot neglect in general the constant $(1/G)^{ab}$. The correct solution is then to consider derivative terms, which automatically eliminate the constant part in the expression for the bracket $[y^a, y^b]$. In the framework of this paper this fact is yet an other indication that the Born-Infeld action is the unique action without derivatives which is invariant under the Seiberg-Witten transformations.

Let us now move to the analysis of derivative terms. To this end we consider a second example, and we compute the double bracket

$$\left[x^a, \left[x^b, x^c\right]\right] .$$

In the flat coordinate system one simply has

$$\left[x^a, \left[x^b, x^c\right]\right] = \left(\frac{1}{g+\omega}\right)^{ad} \partial_d \left(\frac{1}{g+\omega}\right)^{bc} .$$

Using the previous results, and computing in the symplectic coordinate system, one has, on the other hand,

$$\begin{aligned} \left[x_a, \left[y^b, y^c\right]\right] &= -\left(\frac{1}{\partial x_f \partial x_d G^{fd} + B}\right)^{ij} \partial_i x_a \partial_j \left(\frac{1}{G+\Omega}\right)^{bc} \\ &= -\left\{x_a, \left(\frac{1}{G+\Omega}\right)^{bc}\right\} + \{x_a, x_f\} G^{fd} \left\{x_d, \left(\frac{1}{G+\Omega}\right)^{bc}\right\} + \cdots . \end{aligned}$$

Finally, rewriting the result solely in terms of the coordinates y^a, one obtains

$$\begin{aligned} \left[y^a, \left[y^b, y^c\right]\right] &= \left(-G^{ad} + G^{ae}\{x_e, x_f\} G^{fd} + \cdots\right) \left\{x_d, \left(\frac{1}{G+\Omega}\right)^{bc}\right\} \\ &= -\left(\frac{1}{G+\Omega}\right)^{ad} \left\{x_d, \left(\frac{1}{G+\Omega}\right)^{bc}\right\} . \end{aligned}$$

We now see that, recalling the correspondence

$$\{x_a, \longrightarrow \partial_a$$

one has invariance of the term $\left[x^a, \left[x^b, x^c\right]\right]$, up to a minus sign.

Let us now move to a third and last example, by computing the triple bracket

$$\left[\left[x^a, x^b\right], \left[x^c, x^d\right]\right] .$$

As before, in flat coordinates, the expression is easily computed as

$$\left[\left[x^a, x^b\right], \left[x^c, x^d\right]\right] = \left(\frac{1}{g+\omega}\right)^{ef} \partial_e \left(\frac{1}{g+\omega}\right)^{ab} \partial_f \left(\frac{1}{g+\omega}\right)^{cd} .$$

Using again the previous results, and following the usual procedure of expanding the generalized bracket in powers of the induced metric, one obtains

$$
\begin{aligned}
\left[\left[y^a,y^b\right],\left[y^c,y^d\right]\right] &= \left(\frac{1}{\partial x_g \partial x_h G^{gh}+B}\right)^{ij} \partial_i \left(\frac{1}{G+\Omega}\right)^{ab} \partial_j \left(\frac{1}{G+\Omega}\right)^{cd} \\
&= \left\{\left(\frac{1}{G+\Omega}\right)^{ab}, \left(\frac{1}{G+\Omega}\right)^{cd}\right\} + \\
&\quad + \left(\frac{1}{G+\Omega}\right)^{ef} \left\{x_e, \left(\frac{1}{G+\Omega}\right)^{ab}\right\} \left\{x_f, \left(\frac{1}{G+\Omega}\right)^{cd}\right\}.
\end{aligned}
$$

In this case we have complete invariance of the term under consideration. Let us note that a new feature of this last example is that the expression in terms of the variables y^a contains an explicit commutator term $\{(\frac{1}{G+\Omega})^{ab},(\frac{1}{G+\Omega})^{cd}\}$, which is not present in the same expression in terms of the variables x^a. On the other hand the commutator vanishes in the $B\to\infty$, $\theta\to 0$ limit, and therefore should be neglected, as already noted in the definition given in the last section. We are therefore ready to state the following

Claim 2 *Consider a product Π of terms, each of which is a multiple bracket of the coordinate functions x^a (for example one term could be of the form $[[x^a,x^b],[x^c,x^d]]$ or $[x^a,[x^b,x^c]]$). Moreover, let all the free indices $a,b,\dots$ be contracted using the metric tensor g_{ab}. Let us further assume that*

1. *No single term in Π is a single bracket of the form $[x^a,x^b]$ (all terms are derivative terms, so that the constant part in $[y^a,y^b]$ does not spoil invariance).*
2. *There is an even number of basic brackets $[x^a,x^b]$ (which are, by 1, necessarily contained within other brackets).*

   Then the action

   $$\int \Phi\,\Pi$$

   is invariant under the simplified Seiberg-Witten transformations.

Let us note that, given two actions S_1 and S_2 which are invariant, so is the linear combinations aS_1+bS_2. Therefore we should consider in general actions of the form $\int \Phi(a_1\Pi_1 + a_2\Pi_2+\cdots)$, where the coefficients a_i must be determined from other considerations.

6. Some examples

In this last section we analyze in some detail two examples of simple invariant actions with two derivatives, in order to make contact with actions of the form eq. (1.1). We will work only in the flat coordinate system, and we will therefore neglect the index convention, using both indices $a,b,\dots$ and indices $i,j,\dots$

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Let us start by considering the action

$$\int \Phi \; g_{ac}g_{bd}\left[\left[x^a, x^b\right], \left[x^c, x^d\right]\right] .$$

We will for simplicity of notation, but with no loss in generality, take $g_s = 1$ and $g_{ab} = \delta_{ab}$. Moreover we will write all the equations in the case $B_{ab} = 0$, therefore replacing ω with F. The lagrangian then becomes

$$\mathcal{L}_1 = \sqrt{\det(g+F)}\, g_{ac}g_{bd}\left(\frac{1}{g+F}\right)^{ij} \partial_i \left(\frac{1}{g+F}\right)^{ab} \partial_j \left(\frac{1}{g+F}\right)^{cd} .$$

The following quick computation:

$$\begin{aligned} g_{ac}g_{bd}\partial_i \left(\frac{1}{g+F}\right)^{ab} \partial_j \left(\frac{1}{g+F}\right)^{cd} &\simeq \partial_i \left(F - FF\right)_{ab} \partial_j \left(F - FF\right)_{ab} \\ &= \partial_i F_{ab} \partial_j F_{ab} + \partial_i \left(F_{ac}F_{cb}\right) \partial_j \left(F_{ad}F_{db}\right) \end{aligned}$$

can then be used to show that

$$\begin{aligned} \mathcal{L}_1 = &\; \partial_i F_{ab} \partial_i F_{ab} + \frac{1}{4} F_{cd}F_{cd} \partial_i F_{ab} \partial_i F_{ab} + F_{ik}F_{kj} \partial_i F_{ab} \partial_j F_{ab} + \\ & + 2F_{ab}F_{bc}\partial_i F_{cd} \partial_i F_{da} + 2F_{bc}F_{da} \partial_i F_{ab} \partial_i F_{cd} + \cdots , \end{aligned} \tag{6.1}$$

where $\cdots$ denotes terms of order $F^{2n}\partial F \partial F$ with $n \geq 3$.

Let us analyze a second action, again with two derivatives, given by

$$\int \Phi \; g_{ad}g_{be}g_{cf}\left[x^a, \left[x^b, x^c\right]\right]\left[x^d, \left[x^e, x^f\right]\right] .$$

It has the following lagrangian:

$$\mathcal{L}_2 = \sqrt{\det(g+F)}\, g_{ad}g_{be}g_{cf}\left(\frac{1}{g+F}\right)^{ai} \partial_i \left(\frac{1}{g+F}\right)^{bc} \left(\frac{1}{g+F}\right)^{dj} \partial_j \left(\frac{1}{g+F}\right)^{ef} .$$

Again we can expand in powers of F. Using the fact that $(\frac{1}{g+F})^{ab} = (\frac{1}{g-F})^{ba}$ and following the computation:

$$\begin{aligned} &\left(\frac{1}{1-F}\right)^{ia} \left(\frac{1}{1+F}\right)^{aj} \partial_i \left(F - FF\right)_{bc} \partial_j \left(F - FF\right)_{bc} \\ &\quad = \left(1 + FF\right)_{ij} \partial_i \left(F - FF\right)_{ab} \partial_j \left(F - FF\right)_{ab} + \cdots \end{aligned}$$

we conclude that

$$\mathcal{L}_2 = \mathcal{L}_1 + o(F^6 \partial F \partial F) . \tag{6.2}$$

The purpose of the last two examples if twofold. On one hand, they clearly show that the invariant actions introduced in the previous section do have sensible expansions in powers of F and derivatives ∂ and are of the expected general form.

In particular, expressions like eq. (6.1) have already appeared in the literature in various settings [4, 5, 3]. On the other hand the above examples, and in particular eq. (6.2), show that the high F behavior of the action is not completely determined by the first terms in a power series expansion. It is nonetheless true that, given a fixed number of derivatives, the number of possible structures is finite (recall that one cannot introduce non-derivative brackets $[x^a, x^b]$ in the action). Therefore one needs to fix in principle a finite number of coefficients in order to fix completely the high F behavior of the action, given a fixed number of derivatives.

7. Conclusions

We have shown that, with the aid of a generalized Poisson bracket, we are able to construct actions with any number of derivatives which are invariant under a simplified version [7] of the Seiberg-Witten transformations described in [1]. Clearly this is just a first step in a full classification of invariant actions in the sense described in the introductory section.

First of all, even within the present setting, one should show that all invariant actions are linear combinations of the ones described in section 5. We have shown that the actions given in terms of $[\,,]$ are invariant, but by no means we have claimed that all invariant actions are of the form which we have considered. Also, still within the context of the abelian U(1) theory one should reconsider the above analysis using the full group of non-commutative gauge transformations. The results in this paper heavily rely on the geometrically intuitive nature of the simplified transformation $A \to \widehat{A}$ which we have considered. In order to extend the analysis to the more general setting of [1], one should therefore have a better geometrical understanding of the full Seiberg-Witten transformations. This question is probably intimately related to an understanding of the invariance problem in the extremely complex non-abelian case.

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